Plenary Lecture 2

Forty-five Years: An Experiment on Mathematics Teaching Reform

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ABSTRACT This is an experimental report on mathematics teaching reform conducted in an urban-rural fringe area in the west of Shanghai. From 1977 to 2022, this experiment spanning 45 years has witnessed a change of Chinese society from bringing order out of chaos to Reform and Opening-up and eventually to educational modernization. At the early stage of the experiment, a methodological system of practical research, featuring “investigation – screening – experiment – popularization”, was built up, a feasible way to improve the quality of the education under the most common educational conditions was found, and the Chinese experience of “teaching and learning promote each other”, specifically, students learning with experiencing variation and teachers developing through teaching reform, was summed up. At the later stage of the experiment, the focus was shifted to the cultivation of students’ all-round ability and an empirical method of “abductive reasoning through big data” was developed. Based on big data experiments and long-period sampling analysis, by way of clinical observation and evidential reasoning, the key to promote students’ inquiry and innovation ability was found and, with practical effects, it exerted positive influence upon the society. This report is the result of the persistence, perseverance and the collective efforts of three generations, including the life mentors of the older generation, the backbone of the transitional generation and a large number of young talents. The experiment consists of three stages: 1. improve teaching quality generally and dramatically from a low level (1977–1992); 2. comprehend first and break through the bottleneck of high cognition (1992–2007); 3. improve teaching research and promote inquiry and innovation (2007–2022).

Keywords Qingpu teaching reform experiment; Experiencing variation; Action education; Abduction of creative ability.

1. Improve teaching quality generally and dramatically

1.1. A reform experiment initiated by an exam paper

At the end of 1970s, the mathematics teaching quality of the primary and secondary schools in Qingpu County of Shanghai came last in the city. In 1977, we gave a diagnostic examination to 4,373 high school graduates with mathematics problems for

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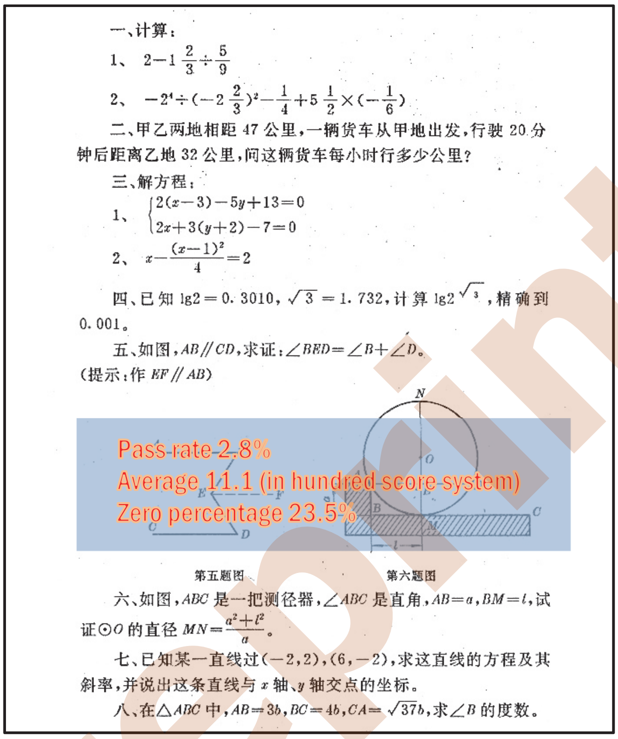
primary school and junior high school students (Fig. 1). The pass rate was only 2.8%, the average score was 11.1 out of 100, and the zero percentage was 23.5%! How should we deal with the shocking backwardness? The tension between students' low score and modern concepts bred an urgent need for teaching reform.



一、计算：

1、 $2-1\frac{2}{3}\div\frac{5}{9}$

2、 $-2^4\div(-2\frac{2}{3})^2-\frac{1}{4}+5\frac{1}{2}\times(-\frac{1}{6})$

二、甲乙两地相距 47 公里，一辆货车从甲地出发，行驶 20 分钟后距离乙地 32 公里，问这辆货车每小时行多少公里？

三、解方程：

1、 $\begin{cases}2(x-3)-5y+13=0\\2x+3(y+2)-7=0\end{cases}$

2、 $x-\frac{(x-1)^2}{4}=2$

四、已知 $\lg 2=0.3010$，$\sqrt{3}=1.732$，计算 $\lg 2^{\sqrt{3}}$，精确到 0.001。

五、如图，$AB//CD$，求证：$\angle BED=\angle B+\angle D$。（提示：作 $EF//AB$）

Pass rate 2.8%
Average 11.1 (in hundred score system)
Zero percentage 23.5%

第五题图 第六题图

六、如图，ABC 是一把测径器，$\angle ABC$ 是直角，$AB=a$，$BM=l$，试证⊙O 的直径 $MN=\frac{a^2+l^2}{a}$。

七、已知某一直线过$(-2,2)$，$(6,-2)$，求这直线的方程及其斜率，并说出这条直线与 x 轴、y 轴交点的坐标。

八、在△ABC 中，$AB=3b$，$BC=4b$，$CA=\sqrt{37}b$，求$\angle B$ 的度数。

Fig. 1. The examination[2] of 4,373 high school graduates in 1977

The backbone teachers in Qingpu pledged to the then director of the bureau of education that despite whatever hardships, they were determined to change the backwardness completely. With a strong determination to avenge the shame, they drew

[2] The English translation of the examination paper:

1. Calculate (the question items omitted).
2. The distance between Place A and Place B is 47 kilometers. A truck starts from place A, and, after 20 minutes, it is 32 kilometers away from place B. How many kilometers does this truck travel per hour?
3. Solve equation or system of equations (the question items omitted).
4. Given $\lg 2 = 0.3010$ and $\sqrt{3} = 1.732$, calculate $\lg 2^{\sqrt{3}}$, accurate to 0.001.
5. As shown in the figure, $AB//CD$, verify that $\angle BED = \angle B + \angle D$. (Hint: Draw EF with $EF // AB$.)
6. As shown in the figure, ABC is an instrument for measuring the diameter of a circle in which $\angle ABC$ is a right angle. If $AB = a, BM = l$, show that the diameter of $\odot O$ is $MN = \frac{a^2+l^2}{a}$.
7. A straight line passes through points $(-2, 2)$ and $(6, -2)$. find its equation and slope. Find also the coordinates of the intersections of the line with the x-axis and the y-axis.
8. In $\triangle ABC$, $AB = 3b$, $BC = 4b$, and $CA = \sqrt{37}b$. Calculate the degree of $\angle B$.

up a plan consisting of three-year all-round investigation, one-year experiences screening, three-year experimental practice and eight-year promotion and application.

1.2. A reform design — "15 years to sharpen a sword"

Teaching reform was a game between change and inheritance. Our design principles for the reform were: firstly, adhere to openness and inclusiveness and absorb extensively the information from outside, including experience regeneration, theoretical progress and technology update; secondly, base our screening on practice, and, with continuous improvement and repetitive testing of teaching effect, look for appropriate path in view of the current situation.

The teaching practice was made up of four parts:

(1) Teaching investigation (three years). Conduct a comprehensive investigation into the major problems and their crux in local area, and we collected more than 160 specific experiences accordingly.

(2) Experiences screening (one and half years). Applying experiences in classroom teaching, we did 50 circular screenings repetitively, which included experiments, observation, filtering and optimizing. With a focus on the improvement of classroom teaching, we came to four key teaching behaviors: i) facilitate students to learn under urgent demand; ii) organize the sequence of teaching content; iii) guide students to attempt and explore; and iv) provide feedback and adjustment on the basis of learning results.

(3) Contrastive experiments (three years). Select 440 junior high school students from ten classes and, following the four aspects mentioned above, conduct contrastive experiments under strict control, track and compare 50 pairs of matched students closely.

(4) Popularization and reinforcement (eight years). Popularize and reinforce experiences in the entire district, and introduce the concept of communication into popularization so that popularization would lead to internalization and creativity instead of mere imitation or copy. Make a full coverage relying on multi-level teaching research groups. What we want is not tinkering with the problem, but a large-scale improvement.

These four parts make a complete methodological system of the reform experiment (Fig. 2).

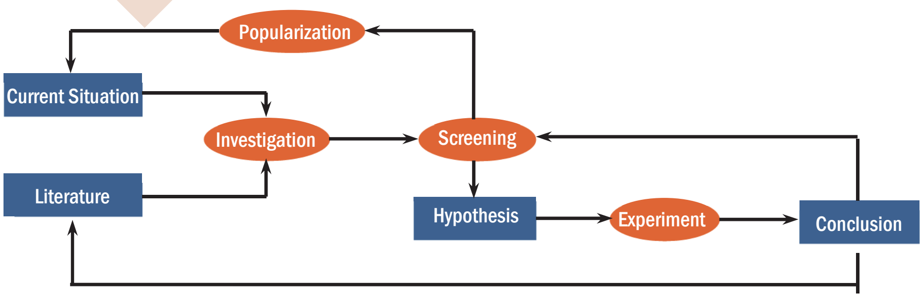


Fig. 2. Method system with circular screening as the core

This experiment always aims at the matching and mutual reflection between full-sample statistics and refined case analysis. We particularly created the method of circular screening, which filled the gap between investigation and hypothesis, and the one between conclusion and popularization & innovation as well. Given our heavy duty of solving real problems, we should avoid the traps of formalization and emptiness, or "scholars revolt" in Chinese idioms.

1.3. Give prompt feedback, make a quick change on the backward situation

The diagnostic examination given in 1977 showed that poor-grade students were all over the county and the total amount was too large to reduce. We took samples from groups with large proportions of poor-grade students, analyzed them individually, observed their homework, and eventually summed up the formation process of their poor performance (1982) (Fig. 3).

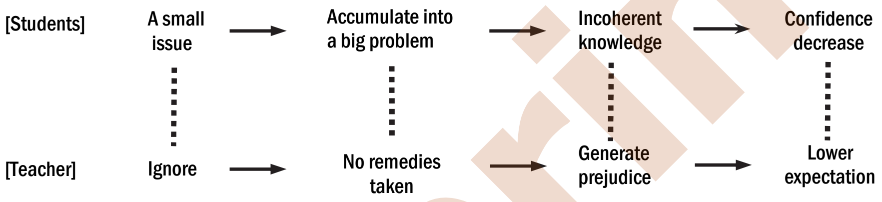


Fig. 3. The formation process of lower-scored students

The observation of students' homework showed that the cause of low score was the accumulation of both teaching and learning problems, which went all the way down to the loss of self-confidence on students' part. We should block this ill cycle promptly. Therefore, we referred to local experiences and gave feedback to students about their assignments within the day, marked their exercises face-to-face and encouraged students generously. Later, we conducted experiments in different groups in 1986, and the result showed that timely and individual guidance with targeted feedback was undoubtedly effective in improving students' score (Fig. 4).

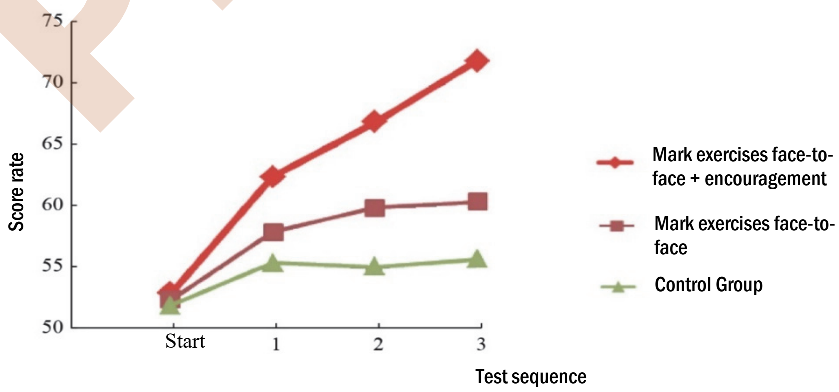


Fig. 4. Experiment results of assignment feedback

As a matter of fact, the *Book of Rites* from ancient China already described a dual-channel feedback model that goes "learn – discovers ignorance – self-reflection" and "teach – discovers difficulty – self-improvement". The module is referred to as "teaching and learning promote each other". It is a comprehensive demonstration of the charm of outstanding ancient Chinese culture (Fig. 5).

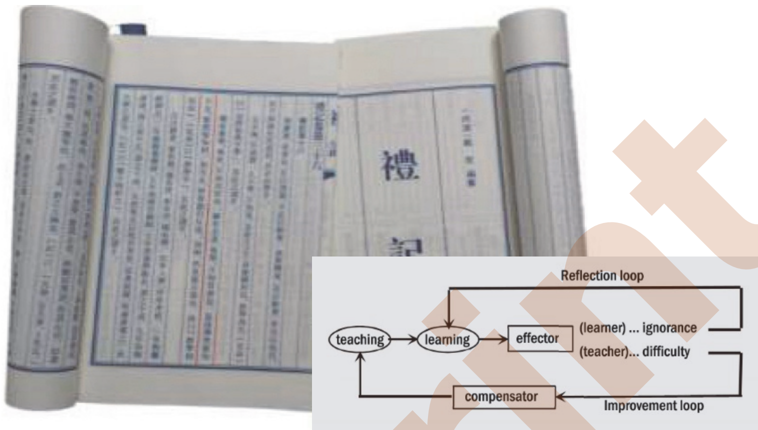


Fig. 5. The dual-channel feedback of "teaching and learning promote each other" in the *Book of Rites*

1.4. *Assessment of mathematical thinking, leading to the deepening and innovation of teaching reform*

In 1984, we used multi-media audio-visual technologies of "think aloud" to assess 50 pairs of students selected from control group and experimental group about their thinking process in solving a problem (Fig. 6). Individual case studies turned out that "experiencing variation" teaching was effective in improving the accuracy, agility and profundity of students' thinking, and this played a critical role in the innovation and extensive promotion at the second stage.

1.5. *From experimental methods to the interpretation of teaching principles*

As of 1992, the methodological characteristics of Qingpu practical educational research could be summarized as the follow:

(1) Scientific discovery models including agreement, discrepancy, covariation, deduction and induction were introduced so as to make the improvement of teaching behavior synchronize with critical thinking. As a result, the possibility and expectations of applying practical research in theoretical innovation was increased.

(2) Practice was decided as the screening method for Qingpu Experiment, by which the gap between investigation and experimental hypothesis was filled up. Taking this as the core, a comprehensive system with multiple methods complementing each

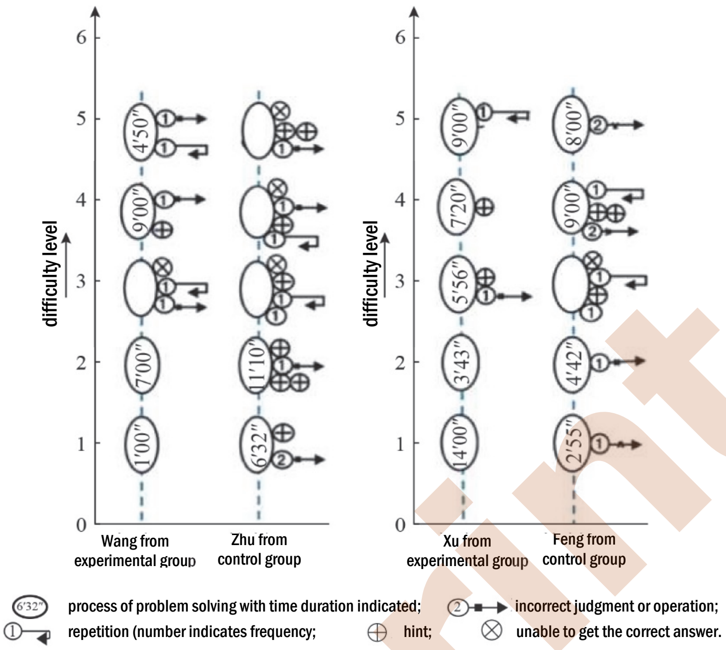


Fig. 6. The Comparison of pair students on their problem-solving processes

educational problems.

(3) Make an in-depth study on the nature and form of the popularization of teaching experiences and explore the condition, form and effect of the popularization of research results in the hope of providing a reasonable basis for the application of teaching reform results in reality and guaranteeing the effectiveness of it as well.

(4) other was constructed, increasing the ability of practical research in solving real

(5) Establish a scientific research community involving researchers and teachers. Setting this community as the subject of the experiment could bring the research into the deep area of actual teaching process and increase the practical effectiveness and socialization degree of teaching reform experiments.

As a matter of fact, this is a comprehensive research based on teaching practice. It is noteworthy that Qingpu Experiment also provided interpretations for the key behaviors of teaching reform practice by having discussions, theoretically, on the process of mathematical cognition and activity, mathematical thinking and target classification, and gradually came out with a series of fundamental principles including Affection, progression, attempt, and feedback.

1.6. Win back quality

After ten years of efforts, the math score of the 9th graders in the entire QingPu County had been increasing year by year (Fig. 7), and the pass rate went from 16% in 1979

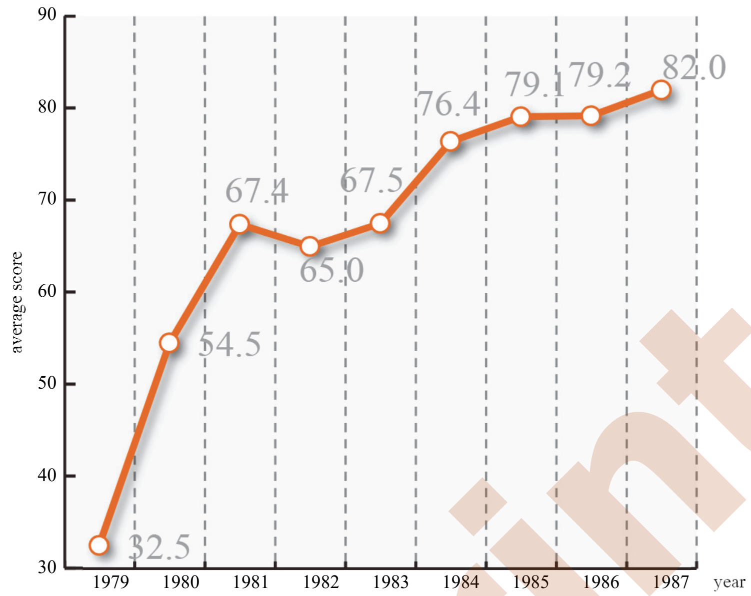


Fig. 7. Scores on the municipal tests of mathematical achievement for junior middle school graduates

(the lowest in the city) to 85% in 1986 (the average rate was 68%). In April, 1986, Shanghai Education Committee convened a general meeting to popularize QingPu experiences in the entire city of Shanghai. In October, 1990, the Department of Basic Education of the Ministry of Education dispatched a research group consisting of 18 experts to Qingpu and conducted a 9-day investigation on the experiences, achievements and practical effectiveness of the Experiment. In 1992, the Ministry of Education designated Qingpu experiences as a major achievement of basic education reform, and formally popularize it in the country.

Qingpu Experiment finally worked out a feasible way to improve teaching quality under the most common educational condition. The major research result of this stage are: *Learn to Teach* (1991) and *Theory of Teaching Experiments — A Study of the Methods and Pedagogical Principles of Qingpu Experiment* (1994).

2. Break through the Bottleneck of High Cognition

2.1. Attach importance to the assessment of students' math cognitive ability

The experiment at the second stage is carried out with two themes: first, the assessment of cognitive ability, and second, dig deep for outstanding experiences. With regard to the first theme, Qingpu Experiment had completed three tests and relevant studies on the math cognitive ability of local eighth graders since 1990s, conducted by professional researchers who designed the tests catering to different abilities respectively (last for 28 years with 840,000 standard statistical data).

The first test was conducted in 1990. In light of Bloom and Wilson's framework, cognitive ability objectives were classified into seven categories: computation, knowledge, comprehension, application, analysis, synthesis and evaluation, and thereafter a test with 50 test items involving 106 knowledge points was designed. The test was divided into three sections with a total of 220 minutes. Altogether 3200 8th graders took the test in paper-and-pencil format with 25 students in a group. Factor analysis technique was used for the analysis of the data. The main conclusions were:

(1) Memorizing and understanding were identified as the two most basic latent factors, and all the seven categories can be expressed through various loadings on the two factors. According to the data, the continuity and equidistance among the categories were not reasonable enough; computation and knowledge were mixed together, comprehension and application, analysis and synthesis could be merged and simplified (Fig. 8).

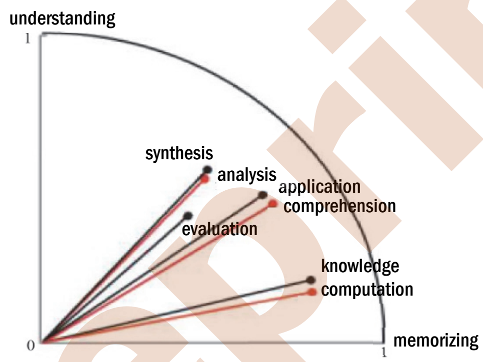


Fig. 8. Factor loadings on memorizing and understanding

(2) The distribution pattern of students' ability tendency was basically clear. Generally speaking, students had a good mastery of computation and concepts, but scored relatively low in comprehension and inquiry (Fig. 9). That is to say, in order to make students "smarter", we must get rid of rote learning and drill practice so as to break the bottleneck of crammed teaching.

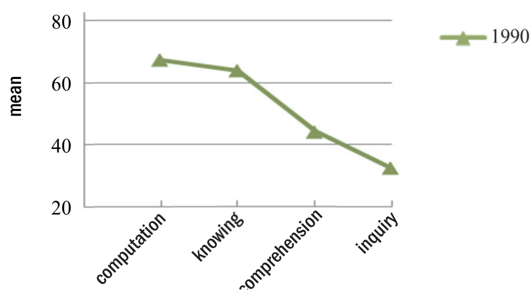


Fig. 9. Students scored relatively low in comprehension and inquiry understanding

2.2. *What if the scores are up while the students are not smart?*

The second theme was, in light of the major goal, digging deep for outstanding experiences. Having reviewed the two-way factorial design of "students attempt under the guidance of teachers + teachers provide timely feedback" (1982–1983), we primarily adopted "experiencing variation" for the practice of "students attempt under the guidance of teachers". The follow is the result of the experiment (Fig. 10).

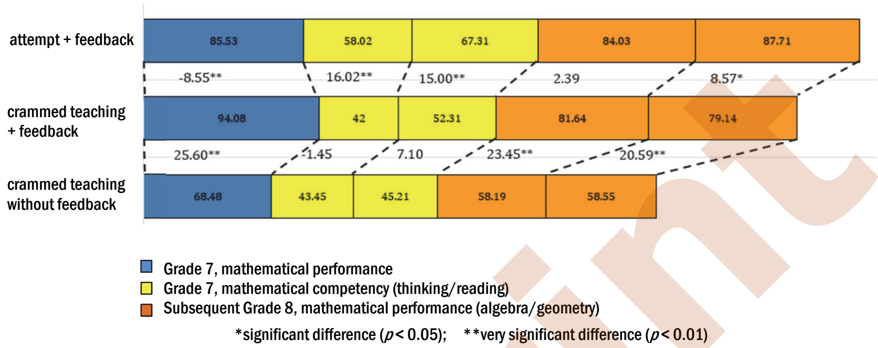


Fig. 10. Experimental results on factorial design by attempt and feedback

(1) "Mastery Learning" with timely feedback accounts for the substantial improvement of grades;

(2) "Crammed teaching + feedback" facilitates neither mathematical thinking nor reading ability and, furthermore, it even affects follow-up learning;

(3) "Attempt + feedback" could make students smarter. "It is more reasonable and helpful for students' long-term learning than the so-called mastery learning," Professor Fonian Liu, a former president of ECNU, said in 1985.

Therefore, "experiencing variation" became a hot topic for classroom teachers at that time. In a paper submitted to Shanghai Mathematical Society in 1981 (Fig. 11), it was clearly noted that variation has two types: conceptual variation and procedural variation, and the latter one is exactly an outstanding traditional experience in solving math problems.

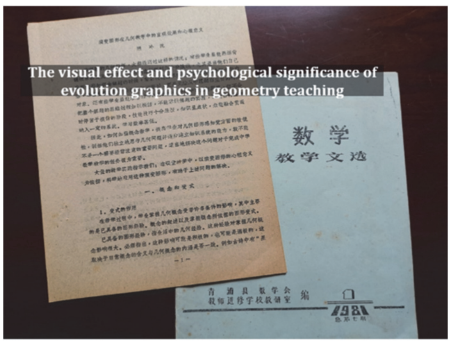
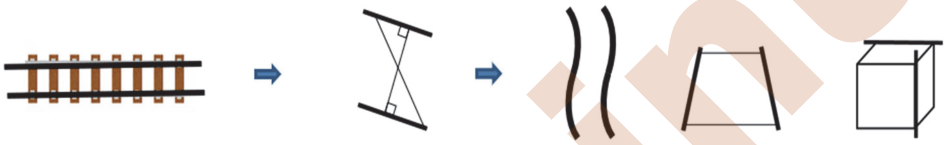


Fig. 11. Copy of a mimeograph paper on variation printed in February1981

2.3. *Deepen understanding in "attempt and experience" (mathematical conceptual variation)*

For example, a simple concept – parallel lines. A teacher told the students that "parallel lines are straight lines that do not intersect with each other in a plane (abstract concept)". A student said, "I've memorized it". This is crammed teaching. Another teacher said, "Parallel lines are like two tracks of a train (descriptive definition)". A student asked, "Are they still parallel lines if the train makes a turn?"

Qingpu Experiment adopted another approach – experiencing variation, which includes three steps: step 1 is concrete and intuitive, they are parallel lines; step 2 is abstract and varied, they are also parallel lines; step 3 is plausible, they are not parallel lines (Fig. 12).



"is" (concrete, virsualized) "is also" (abstract, varied) "is not" (apparently "is" but actually "is not")

Fig. 12. Method of experiencing variation

Many experiments showed that: constructing mathematical concepts through the variation of material or form, from concrete to abstract, and via experiences of "is", "is also" and "is not", was able to i) reduce students' cognitive burden significantly; ii) deepen students' understanding of the key properties of mathematical concepts; and iii) improve independent identification ability in deceptive scenes.

2.4. *Hold onto "core connection" while solving math problems (procedural variation)*

Procedural variation involves meticulously designing "Pudian" （scaffolding） for mathematical problems to go from easy to difficult, or reducing problems from complex to simple, namely "simplification". Hence, students are guided to solve problems by themselves without mechanical practice.

In 2016, we reexamined the experimental materials of "Activity Process Analysis" and studied the connection patterns involved. For example, the application question of "a truck crossing a bridge" for 7th graders usually consists of two parts: i) When does the truck start to drive onto the bridge? ii) When does the truck start to move on the bridge? These are two critical points, which could be expressed through an "external" static line segment diagram and an "internal" one. If we make the truck move from left to right, three more questions could be asked: i) When is the truck not on the bridge yet? ii) When is the truck in the process of going onto the bridge? iii) When is the truck moving on the bridge? Holding onto the "core connection" of "moving line diagram", students could be successfully transferred to the discussion of the five kinds of relationships between two circles for 9th graders (Fig. 13).

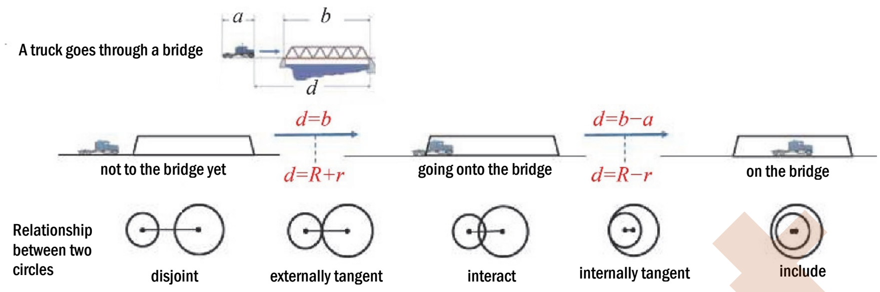


Fig. 13. Connection between truck going through the bridge and the relationship between two circles

The experiment data showed that when students applied what they had already known in solving new problems, it was important for them to find the most essential and transferrable element — the core connection. This connection could i) shorten the cognitive distance between the new problem and the anchoring point of existing knowledge; ii) significantly improve the degree of transfer in learning process; and iii) stimulate students' constructive thinking in solving math problems.

2.5. Chinese experience of "experiencing variation"

Teaching via "experiencing variation" focuses on students' conceptual understanding of mathematics and their constructive thinking in solving problems. Early experiments show that it can advance the transition of secondary students' mathematical thinking from visualization-based judgment to logical reasoning at least one year earlier (Fig. 14). After extensive practice, experiments and improvement, it has become a well-known Chinese experience.

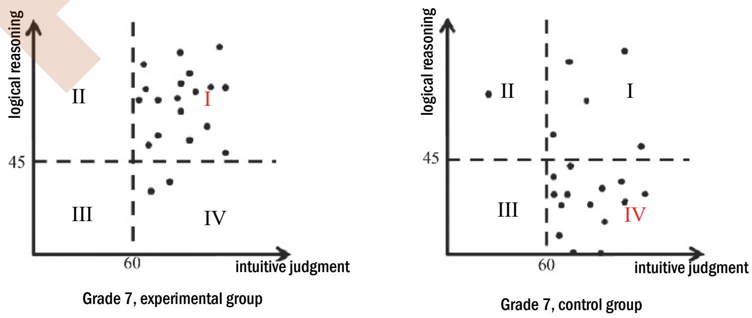


Fig. 14. Comparison of students' mathematical thinking

Variations can be applied in improving teaching via three dimensions: logical reasoning, situational application, and learning psychology. Two issues should be noted: first, that the more variations the better is not true. Rather, variations should be designed in accordance with the goals and needs of various student groups. Second, the key to variations is “experience”. They are not crammed teaching in a disguised form. Instead, students should be given more opportunities to participate, attempt and express their ideas.

2.6. The change after more than ten years

The second test was conducted in 2007. With the improvement of classification of learning objectives (such as distinguishing knowledge from cognition, simplifying cognitive objectives, paying attention to creative and inquiry thinking), the test items were adjusted according to the reality of local students. The duration of the test remained unchanged, and 4349 students took the test. This time the accuracy of factor analysis improved significantly. The loadings of each cognitive objectives on the two main factors accounted for 85.15% of the total variance, which was 24% higher than that in 1990. The main conclusions are:

(1) A four-level structure, which was relatively concise and in line with the regional reality, was constructed, aiming to promote in-depth learning and teaching (Fig. 15): computation — rote memorizing; knowing — meaningful memorizing; comprehension — interpretive understanding; inquiry — discovery understanding.

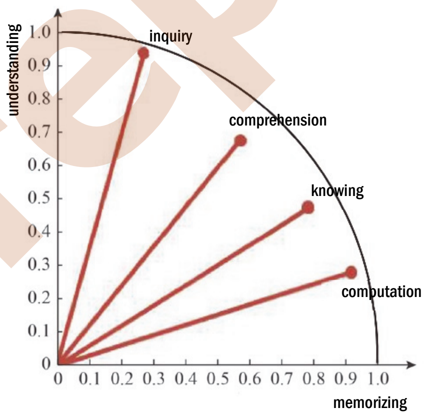


Fig. 15. A relatively simple four-level structure

(2) Compared with 1990, the scores of computation, knowing and comprehension in 2007 had greatly improved. However, the score of inquiry remained unchanged although much effort had been paid. The average score of inquiry was 28.96, which was slightly lower than 32.43 in 1990 (Fig. 16), and it became a new difficult point needed to be dealt with.

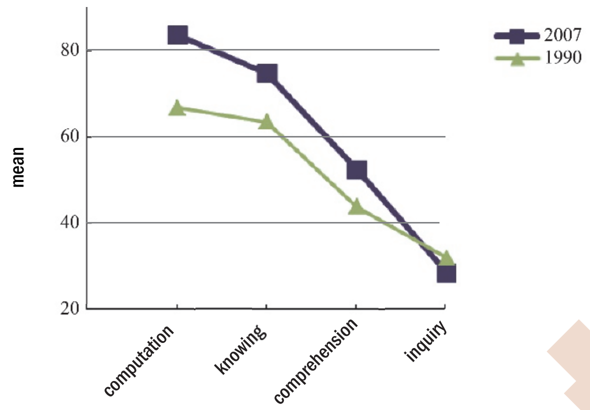


Fig. 16. The score of inquiry remaining unchanged

The major research results of this stage are: *Action and Interpretation of Teaching Reform* (2003); *A Witness of Reform — Lingyu Gu and Thirty-year Qingpu Teaching Experiment* (2008).

3. Promote Inquiry and Innovation

3.1. What if inquiry and innovation abilities remain unchanged?

Before entering the 3rd stage, we'd better take a look at the third test conducted in 2018. This test is exactly the same to the one conducted in 2007 in terms of test items and durations. A total of 3474 students took the test. The main conclusions were as follows.

(1) The average score of comprehension exceeded the passing level, and the score of inquiry was 11.31 percentage points higher. The difficulty was broken through to some extent (Fig. 17).

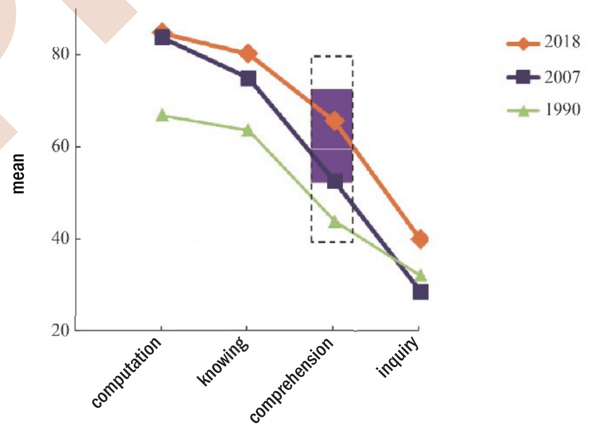


Fig. 17. Results of three tests

(2) The scatter plot showed that there was a positive correlation between comprehension and inquiry. With the improvement of students' comprehension ability, the inquiry level increased exponentially (Fig. 18).

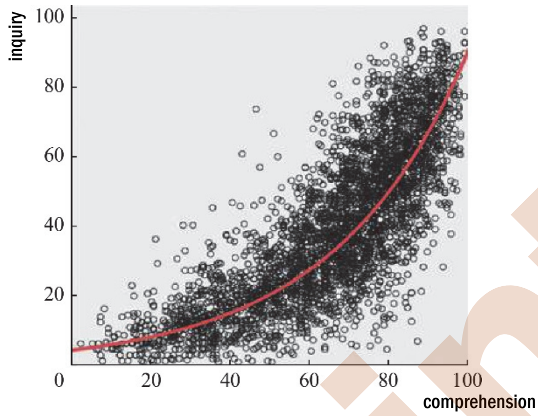


Fig. 18. A positive correlation between comprehension and inquiry

3.2. A turning point

As shown in Fig. 19, we can see the change of the correlation coefficients of comprehension and inquiry. The score of comprehension is between 45 to 95, and the graph is almost two broken lines. The turning point is in the middle. The increasing rate of correlation coefficients on both sides around the turning point shows a cliff style drop, decreasing sharply from 8.4% to 3.2%, indicating that the level of inquiry after the turning point is also affected by other factors besides comprehension.

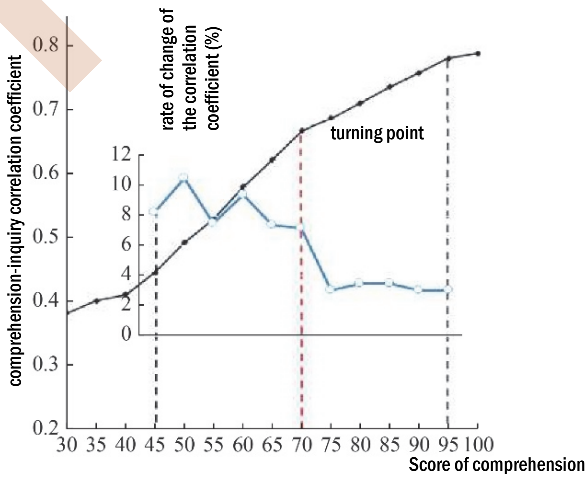


Fig. 19. Turning point of correlation coefficients appearing

Making a 2 by 2 segmentation (Table 1) of the average scores of comprehension and inquiry before and after the turning point, two types of students worthy of attention – type A and type B, were isolated for further analysis. The feature of type A students is that they stick to comprehension while type B students incline to inquiry. The data shows that both before and after the turning point, the number of students in groups of "double high", "double low" and "interaction" was about 1/3 whereas the ratio of type A and type B students within "Interaction" changed from 61:39 before the turning point to 46:54 after the point， indicating the significant influence of inquiry.

Tab. 1. 2 × 2 Classification of different student groups

	Low level of inquiry	**High level of inquiry**
High level of comprehension	Type A interaction: stick to comprehension	Double high
Low level of comprehension	Double low	Type B interaction: go toward inquiry

3.3. Cause-tracing empirical research based on big data

In teaching reform practice, in order to look for critical measures to improve students' inquiry ability, we designed an empirical research method named "cause-tracing". Specifically speaking, we selected typical sample groups available for comparison based on big data, and drew our conclusions through natural observations and evidential reasoning. Research flow chart is as shown in (Fig. 20). This method has the following major features: first, it demonstrates the theoretical penetration in the interpretation of phenomena and causal connections, good for the research group to exert its subjective initiative in applying background knowledge; second, it avoids subjective randomness in observations and sampling, and therefore guarantees the objective strictness in discovering logic. As a result, this method significantly increases the actual contribution of natural observations.

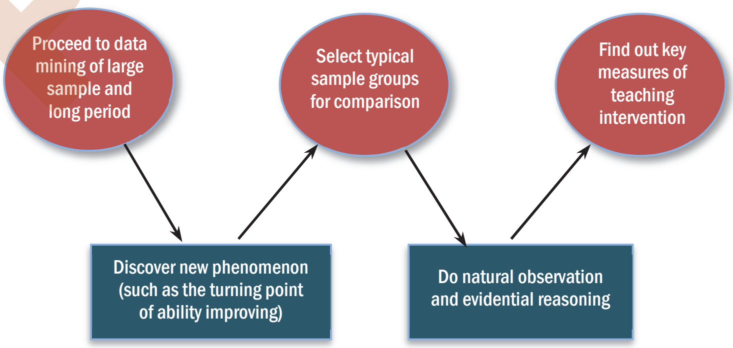


Fig. 20. Research flow chart of cause-tracing

In the research at this stage, we selected a total of 516 students consisting of both type A and B students before and after the turning point for further analysis.

(1) Elaborate data analysis reveals that the cognitive efficiency of type A & B students before and after the turning point is obviously different. They are not of much difference in knowledge understanding and regular application abilities, but type B students demonstrate overwhelming advantage in analysis, judgement, and discovery abilities (Fig. 21).

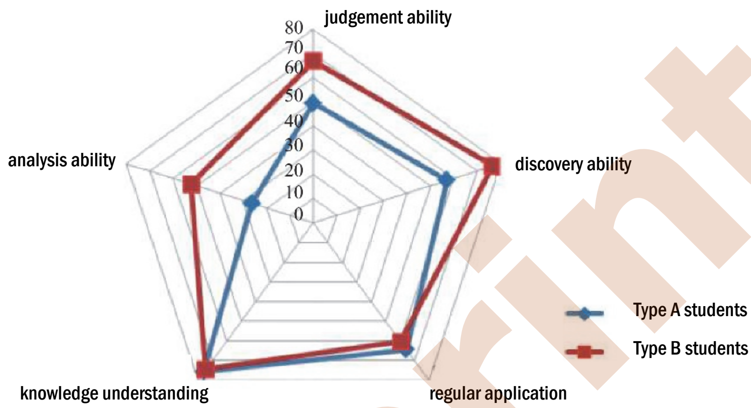


Fig. 21. Different patterns of cognitive ability of the two types of students

(2) Field research made in the schools where these students study reveals that type A students have a strong desire for high scores while type B students tend to spare energy in learning what they are interested in (Tab. 2).

Tab. 2. Characteristics of the two types of students

Type A (Have a strong pursuit for high scores)	Type B (Free up energy for independent study)
Practice common problems repeatedly; familiar with common problems and even reach the degree of "automation"; high accuracy	Dissatisfied with repeated practice; have a strong curiosity; good at asking questions and self-questioning
Good at grasping the requirement of examinations; know their own deficiency in their knowledge and skills, and try their best to make up for them; not interested in inquiry questions	Refuse to follow the majority and have their own opinions; like to solve problems by attempting; have failure experiences
Have a strong pursuit for high scores, sensitive to each single mark	Keep a grade above average; spare energy in solving problems they are interested in
Listen to teachers' command; care for teachers' praise	Not easy to get teachers' attention because they do not obey rules

To conclude, there exist two different types of students, learning methods and results before and after the turning point. It is proved that excessive mechanical training is not advisable. Instead, we must look for key teaching behaviors aimed at improving inquiry ability.

3.4. In-depth clinical observations and comparisons of typical samples

Reasonable math teaching behaviors were not discovered until we went deep into the practical field in which math problems were solved. We wouldn't be able to find key teaching behaviors leading to the improvement of inquiry level until we conducted clinical analysis and comparisons of type A & B students' reaction in the tests and the teaching situations where they are in. The following are some typical teaching cases.

(1) Galileo space amphitheater design (2006)

Experimental schools, where most of type B students could be found, adhere to "activity – development" teaching scenario at a long-term basis. Fig. 22 is an assignment instruction at that time. The design of the space amphitheater is challenging: meet the technical standards and in the meanwhile, accommodate a maximum number of seats. The number of radiation lanes was decided by the designer. The less the lanes, the more the seats, but the number of seats for each row should not exceed thirty. The assignment is not difficult in terms of the knowledge required, but it is very challenging for thinking ability, and students have to deal with dilemmas before working out a solution. Obviously, this is a high-level assignment for students in training their innovative thinking ability.

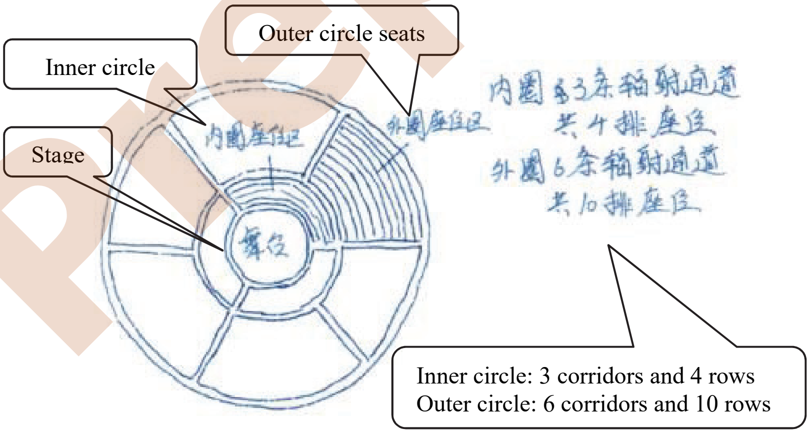


Fig. 22. Students' task: Galileo space amphitheater design

(2) Meticulous thinking ability training for unconventional travel questions (2018)

The follow is a question in inquiry section in one of the three ability tests: Ship A and Ship B set out at the same time from Island I, and shuttle between Island I and Island II. Ship A travels at 10 kilometers per hour and ship B at 8 kilometers per hour, and after 24 hours they return to Island I at the same time. Questions: 1) What is the distance between the two islands? 2) Have the two ships ever reached Island II at the same time?

Though Type A students were familiar with the formula $s = vt$ and the types of questions such as "meet" and "catch up", but they had only arrived an accuracy rate 13.8% when answering to Question 1) due to the demand of careful analysis on "round trip". For Type B students, they thought in this way: Within 24 hours, Ship A travels one more round than Ship B, so

$$s = \frac{24(v_A - v_B)}{2} = \frac{24 \times (10 - 8)}{2} = 24.$$

The accuracy hit 46.2%.

The interview after the test showed that Type B students would question the problem- solving process and further refine it. For example, they would ask, "why one more round for sure?" If we change v_B to 6 kilometers per hour, then $v_A/v_B = 10/6 = 5/3$, Ship A travels 5 rounds and Ship B 3 rounds. They return to Island I at the same time after 24 hours. As a result, ship A travels 2 more rounds than ship B. Generally speaking, the solution is:

$$s = \frac{12(v_A - v_B)}{a - b}, \text{where } \frac{v_A}{v_B} = \frac{a}{b} \left(\text{in which } \frac{a}{b} \text{ is a reduced fraction}\right).$$

For Question 2), most students worked it out by list method. Some Type B students said, "if the two ships ever reach Island II at the same time, it must be 12 hours later, but ship B is not there at that time". This thinking is generated by counter evidence method, but the students can't explain it clearly. Under the guidance of the teacher, they came to the following result:

The time needed for ship A to reach Island II is $2.4m$ ($m = 1, 3, 5, 7, 9$);
The time needed for ship B to reach Island II is $3n$ ($n = 1,3,5, 7$).

Suppose the two ships ever reach Island II at the same time, then $2.4m = 3n$, that is, $4m = 5n$, both m and n are odd numbers. This is impossible. Therefore, the two ships have not reached Island II at the same time.

(3) Interdisciplinary learning and inquiry of the barycenter of a triangle (2018)

This is an in-class inquiry: the barycenter of a triangle — an interdisciplinary problem solving via mechanics and geometry. The mechanics knowledge applied here is primarily using suspension line method to find the barycenter via the balance of particles. The mechanics approach let students understand the consistence of mechanics barycenter with the geometrical conclusion that the three medians of a triangle meet at a point. Students can go further to discuss, via mechanics way, the issue that the three angular bisectors of a triangle or the three heights meet at a point, or more generally to discuss of Ceva Theorem. With closer essential connection between disciplines, there appear many creative methods for problem solving. For example, for an in-class exercise about the barycenter of an isosceles trapezoid, as shown in Fig. 23, $AD = 6,\ BC = 12,$ and $EF = 9$, students came up with several solutions:

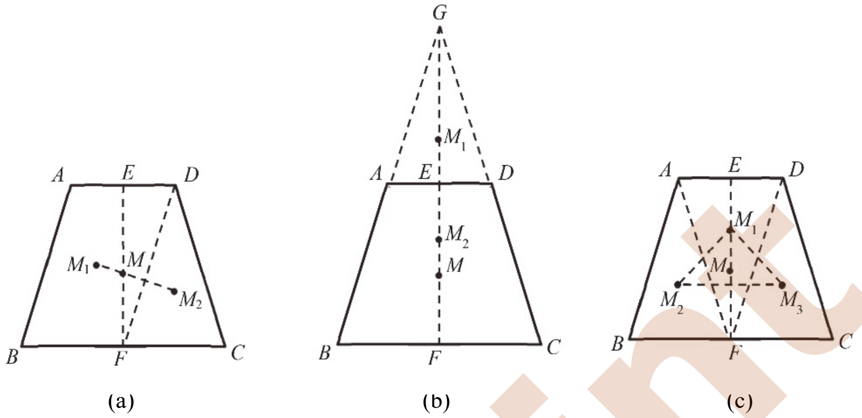


Fig. 23. Barycenter of an isosceles trapezoid

Method 1: intersection plotting

Connect D and F. Denote the barycenter of parallelogram $ABFD$ by M_1 and the barycenter of $\triangle DFC$ by M_2. Line segment M_1M_2 intersects the central axis EF at the barycenter M of the trapezoid, as shown in Fig. 23(a).

Method 2: mechanics equilibrium

Extend BA and CD respectively until they intersect at point G. Let M_1 be the barycenter of $\triangle GAD$ and M_2 the barycenter of $\triangle GBC$. Suppose the barycenter of trapezoid $ABCD$ is M. According to the balance of forces, we got equation $M_1M_2 : M_2M = 3 : 1$, then $M_2M = 2$. The location of M is worked out, as shown in Fig. 23(b).

Method 3: Equivalent reasoning

Divide the trapezoid into three congruent isosceles triangles, and their barycenters are M_1, M_2, and M_3 respectively. The barycenter of $\triangle M_1M_2M_3$ is the barycenter of the trapezoid, as shown in Fig. 23(c).

With regard to the various solutions figured out by the students, the teacher said with great emotion, "the multiple methods the students presented went way beyond my expectation!"

(4) The modeling, evaluation and revision of water hyacinth propagation (2021)

This is a real issue. An aquatic floating plant named water hyacinth grows in the water towns of Qingpu. It reproduces so quickly that it would cover a large area of water in short time and cause trouble to environment, water transportation, drinking water safety for human and livestock and fishery production. In order to discover the reproduction law of water hyacinth, the 8th graders used the knowledge they just learned, namely linear function and line chart, to create a math model of the reproduction of water hyacinth. As a result, they came up with an exponential curve: $g = 25.01 \times 2.16^{t}$, where g stands for the quantity of the plant, 25.01 is the initial value, and t refers to the time at 10-day pace. After a comparison between the model data and the measured data, a big error was revealed in the 3rd month. Then they searched references and consulted professionals, trying to find the cause and make many revisions on the model. Surprisingly, some students figured out an error-trial method by themselves and minimized the sum of absolute values of the errors of each test point. This is equivalent to the thinking of so-called "least absolute deviation". With the preliminary experiences in math modeling and revising models, these middle school students acquired a significant foundation for their future study and research in math modeling.

3.5. Finding key teaching behaviors

A review of the backstepping process mentioned before: first, by refining the test data, we found the completely different cognitive abilities between type A students and type B students; through field research, we summarized the distinctive differences of the expressive characteristics between the two groups of students. Second, our research included both the case studies on the classroom teaching scenario in which type B students are in, and analysis and comparison of the two groups of students' reactions in the tests. This is clinical observations and studies under the guidance of data. In this way, we traced back gradually until we found the following key teaching behaviors that were effective for improving students' inquiry ability:

(1) High-level task-driven teaching design

- Organize challenging teaching tasks according to the internal hierarchical and sequential structure of knowledge;
- Pre-design individual targets according to different needs of students;
- Build comfortable and energetic learning environment and encourage learning with high-level concentration.

(2) An independent study process with fine processing of thinking

- With multiple strategies, activate previous experience and connect it with exploratory tasks;
- Through application, experiencing, or and knowledge assimilation, make fine processing of thinking;
- Make constant feedback and revision on reprocessing learning upon the evaluation of thinking effectiveness.

3.6. Teaching research follow up, attach importance to inquiry learning

Going from comprehension to inquiry, teaching research is indispensable. Based on the case studies of a number of master teachers from Jiangsu Province, Zhejiang Province and Shanghai, particularly Madam Yu Yi's experience of "Prepare one lesson three times, keep doing this for three years, and you're bound to be a good teacher", Qingpu Experiment proposed an in-service professional development mode called "Action Education" in 2004 (Fig. 24). It includes three focuses (focus on personal experience accumulation, focus on new ideas and experiences, and focus on the real gains of students), two reflection holders (looking for the gap between oneself and the others, looking for the gap between plan and reality), and one carrier — *Keli* (Exemplary Lesson Development). They make a cooperation platform on which all these elements repeat alternately, indicating the unique organizational culture and action route of teaching research in China.

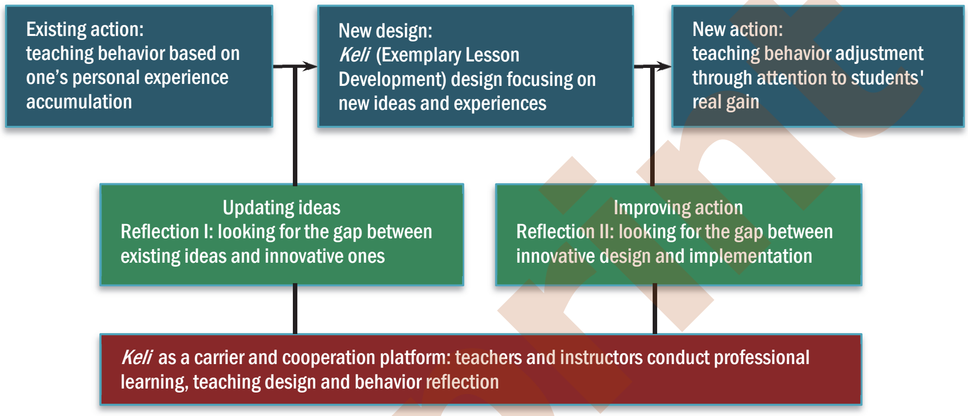


Fig. 24. The model of action research

Some scholars proposed knowledge sharing model of interpersonal learning and developed two skills: first, "expose oneself" (verbal guidance and analysis aimed at exposing real problems); second, "listen and respond" (reflective absorption and response skills). Due to the development of some high-level inquiry learning models such as discovery learning, practice learning and project learning, and so on over the years, teachers usually found themselves in an unknown area called "public issues area". Therefore, besides "knowledge sharing (knowing and not knowing)", the new concept of "create teaching behavior collaboratively (yes or no)" became particularly important. Qingpu Experiment summed up the following two key skills: first, "problem sorting" (sort out the problems about the inquiry target and create "public problems area"); second, "design and improvement" (design inquiry process and make repeated revisions based on evidences). In this way, we came up with two models of interpersonal learning (Fig. 25).

Tab. 3 is a lesson about "introduction of irrational numbers" in 2015. After three times circulation going from design to improvement, it eventually succeeded in making students learn math via "doing math". Students' learning style was successfully changed.

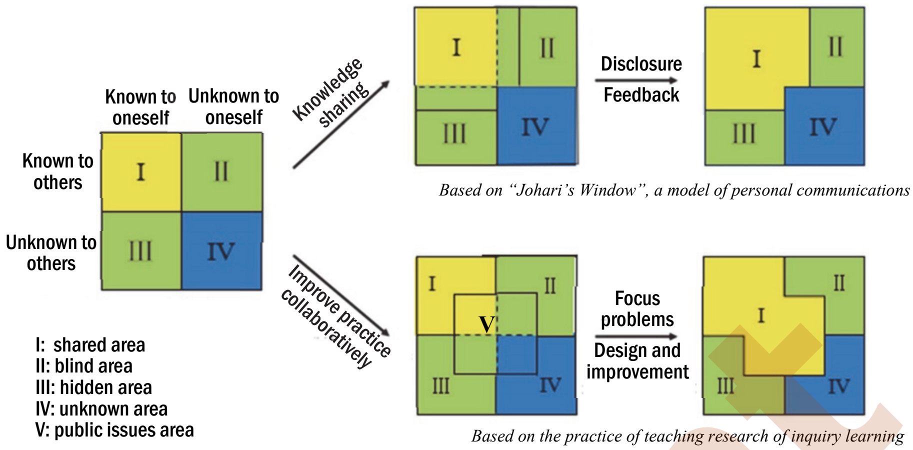


Fig. 25. Two models of interpersonal learning

Tab. 3. Lesson revision based on video analysis

Initial design	Issues during implementation	Revision
The first round — light humanistic purport		
Use historical stories about irrational numbers to arouse students' interest and experience scientific spirit.	Restricted to story plot with the mathematical discovery process left out. Have to return to the old way of passive explanation.	Stories settle in to clarify the cause of the extension of the concept of numbers. Use "$\sqrt{2} =$?" to kick off the replay of the discovery process.
The second round — attend to mathematical process		
Use decimals to approximate, and through deduction, identify the key attribute that irrational numbers cannot be represented as fractions.	Students understood the calculation and deduction. Nevertheless, they still asked: is $\sqrt{2} = 1.4142\ldots$ a definite number while it is growing all the time?	Look for the origin of the issue and choose an appropriate analogy: $\frac{10}{3} = 3.3333.$ It grows all the time, but it is a definite number, implying the idea of limit.
The third round — change the learning mode		
To avoid mere deduction and explanation, design worksheets on $\frac{10}{3}$ and $\sqrt{2}$ respectively, requiring students to discuss while doing math.	Deepen the experience of "gradual approximation" in doing and discussing math. The degree of understanding varies from student to student.	The worksheets and discussion questions could be designed into a certain gradient, so as to cater to the need of different students.

"Teachers become practitioners with research ability" is the pursuit and expectation of Qingpu Experiment in educating outstanding teachers.

(1) At early stage, we required backbone teachers to sit in on the class and make observations. Not a single class should be missed. At that exact moment and inspired by what they saw, teachers thought about the advantages and disadvantages of the lesson and worked out methods for improvement. This strategy was very effective in training teachers' teaching sensitivity. But this kind of observation was restricted by

incompleteness of information and the limitation of memory.

(2) Since the end of 1990s, the introduction of information technologies, including video tapes at the beginning and CD and other video facilities later on, has witnessed breakthroughs one after another in our research. Multi-angle holographic recording made it possible to collect a great deal of data for analysis; playback, freeze and "microscopic" made it possible to study fleeting subtle behavior (including verbal expression, action and facial expression). Among these, the change of teachers' role — teachers became "both the actors and audience", was the centerpiece that provided a convenient way to educate a great quantity of good teachers.

(3) Nowadays, many teachers are becoming "cycle improvers" because the diversity of evidences has been greatly expanded by video analysis, the resource of data statistics is unprecedentedly abundant, and the materials for case analysis are much more refined. In the process of teaching research, these serve as a foundation for some people to make empirical interpretation, and for theorists to make rational deduction. Cycle improvement based on multiple evidences that prove each other pushes "Teachers become practitioners with research ability" to a higher step.

The major research results of this stage are: *Action Education – A Paradigm Innovation on Teachers' In-service Education* (2007); and *Oral Teaching Reform — Regional Experiments or Research Chronicle* (2014).

4. Concluding Remarks

Based on classroom experiments, our reform started from a low point in the hope of improving teaching quality, and gradually developed into a goal-oriented practice with different stages. We broke cramming via "experiencing variation" and promoted inquiry-based mathematics learning by designing "high-level tasks". Nevertheless, we're still on the way of reform given the big change in the current era, and therefore, negligence and mistakes are unavoidable. Criticism and suggestions are welcome.

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